QUESTIONS AND DISCUSSIONS.

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NEW QUESTIONS.

43. Is any rapid special method known for the evaluation of the Sylvestrian determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape, and of its frequent occurrence; yet no text that I am familiar with deals with this form in detail. Will some colleague who is a specialist in this field and knows the periodical literature give us some information on this matter?

44. Is there any known formula for the co-factor of the element a_{ij} in the "binomial" determinant here shown?

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ C_1{}^2 & 1 & 0 & \cdots & 0 & 0 \\ C_2{}^4 & C_1{}^4 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{n+1}{}^{2n} & C_n{}^{2n} & C_{n-1}{}^{2n} & \cdots & C_1{}^{2n} & 1 \end{vmatrix}$$

Calling the co-factors A_{ij}, it is plain that $A_{ii} = 1$, and $A_{ij} = 0$, $(i > j)$. The co-factors of the zero elements, $(i < j)$, are undoubtedly expressible in a formula; the writer would like to know if any reader of the MONTHLY has met with such a formula anywhere.

DISCUSSIONS.

Professor MacNeish shows how the intersections of two conic sections with given foci and directrices may be found by the use of ruler and compass, in case they have a focus in common. The problem reduces to special or to limiting cases of the problem of Apollonius.

Perpetual calendars have been discussed in the MONTHLY by Roman [*1915*, 241] and Morris [*1921*, 127]. Mr. Franklin calls attention in the second discussion below to a formula obtained by Zeller enabling one to calculate directly the day of the week on which any date will fall.

I. THE INTERSECTIONS OF TWO CONIC SECTIONS WITH A COMMON FOCUS.

By H. F. MACNEISH, College of the City of New York.

The determination of the points of intersection of two conic sections having given foci and directrices is in general a problem of the fourth degree, and cannot be solved by ruler and compass. If, however, the two conics have a common focus the problem reduces to one of the second degree. The geometric solution

is here made to depend on the Problem of Apollonius[1]: To construct a circle tangent to three given circles (in particular cases the circles may reduce to points or lines).

The problem may be applied to the determination of the intersection of orbits in astronomy in the case of ellipses and parabolas; and to the determination of the location of artillery by the sound of the discharge in the case of hyperbolas.[2]

Case 1. Two Parabolas. Draw a circle passing through the common focus F of the parabolas, and tangent to the two directrices. There will be two such circles; their centers C_1, C_2 will be the intersections of the parabolas.

Proof. C_1, C_2 are both equally distant from the focus and the directrix of each parabola.

Case 2. Parabola and Ellipse. Let the common focus be F and the other focus of the ellipse E. With E as center and radius equal to m, the major axis[3] of the ellipse, describe a circle E. Construct circles each passing through F and each tangent to E and to the directrix of the parabola. Their centers C_1, C_2 will be the intersections desired.

Proof. C_1 and C_2 lie on the parabola, since they are equidistant from F and the directrix. Since the line EC_1 passes through T_1, the point of tangency of circles E and C_1, $EC_1 + C_1F = EC_1 + C_1T_1 = ET_1 = m$; therefore C_1 lies on the ellipse; a similar proof holds for C_2. There will be either two points of intersection which may coincide or none.

Case 3. Parabola and Hyperbola. Let the common focus be F and the other focus of the hyperbola H. With H as center and radius equal to t, the transverse axis[3] of the hyperbola, describe a circle H. Construct circles each passing through F and each tangent to H and to the directrix of the parabola. Their centers will be the intersections desired.

Proof. Similar to the preceding case. There will in general be four intersections.

Case 4. Two Ellipses. Let F be the common focus, E and E' the other foci, m and m' the corresponding major axes. With centers E and E' and radii m and m' respectively draw circles E and E'. The point F is interior to both circles. Construct circles through F tangent to E and E'. Their centers are the intersections of the ellipses.

Proof. Similar to case 2. There are in general two intersections.

Case 5. Ellipse and Hyperbola. Let F be the common focus, E the other focus and m the major axis of the ellipse, H the other focus and t the transverse axis of the hyperbola. Draw a circle E with center E and radius m, and a circle H with center H and radius t. Construct circles through F tangent to H and E. Their centers will be the intersections of the conics.

Proof. Let one of the circles found, C_1, with center C_1 be tangent to E at T_1

[1] Solutions of some of the cases are indicated in *Methods and theories for the solution of problems of geometrical construction*, by J. Petersen, Copenhagen, 1879, problems 181, 187, 238, 276, 401, 402, 403.

[2] See H. F. MacNeish, *School Science and Mathematics*, October, 1918, p. 626. Cf. *1921*, 36.

[3] This length may easily be constructed, since foci and directrices are given.

and to H at T_1'. Then

$$EC_1 + C_1F = EC_1 + C_1T_1 = ET_1 = m,$$
$$HC_1 - C_1F = HC_1 - C_1T_1' = HT_1' = t;$$

and C_1 lies on both curves. There are in general four such intersections.

Case 6. Two Hyperbolas. The construction and proof in this case are entirely similar to those of Cases 4 and 5. There are in general four intersections.

II. An Arithmetical Perpetual Calendar.

By Philip Franklin, Princeton University.

In connection with the discussion of perpetual calendars given by Doctor Morris in the Monthly, *1921*, 127, attention is called to a formula given by Christopher Zeller[1] which enables one to obtain by merely arithmetical operations the day of the week on which any given date falls. It is thus an arithmetical perpetual calendar, giving the same information as the mechanical ones described by Doctor Morris.

$$w = \left[\frac{c}{4}\right] - 2c + \left[\frac{y}{4}\right] + y + \left[\frac{(m+1)26}{10}\right] + d,$$

where c is the number of the century,
y is the number of the year in the century,
m is the number of the month,[2]
d is the day of the month,

and the number of the day of the week to be found is the remainder obtained by dividing w by 7. $[X]$ means the greatest integer in X.

E.g., for March 4, 1921, we have

$$w = \left[\frac{19}{4}\right] - 2 \times 19 + \left[\frac{21}{4}\right] + 21 + \left[\frac{(3+1)26}{10}\right] + 4$$
$$= 4 - 38 + 5 + 21 + 10 + 4 \equiv 6 \pmod{7},$$

giving the sixth day of the week, or Friday; while for February 22, 1921, we have

$$w = \left[\frac{19}{4}\right] - 2 \times 19 + \left[\frac{20}{4}\right] + 20 + \left[\frac{(14+1)26}{10}\right] + 22$$
$$= 4 - 38 + 5 + 20 + 39 + 22 \equiv 3 \pmod{7},$$

giving the third day of the week, or Tuesday.

As noted by Zeller, to prove the formula correct, we have merely to check it for one date and notice that it gives the proper changes when we increase any of the numbers on which it depends. The reader will find this statement easy to verify if he uses the facts given in the article referred to above.

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[1] *Acta Mathematica*, vol. 9, 1887, pp. 131 f.

[2] January and February are counted as the 13th and 14th months of the preceding year.